A comparison between analytical and numerical solutions of the Vlasov equation

Juan R. Sanmartin

E.T.S.I.Aeronáuticos
Universidad Politécnica de Madrid

XVI Congreso de Métodos Numéricos y Aplicaciones
ENIEF

Córdoba – Argentina, Octubre 2007

<u>Work motivation</u>:

Relative validation of PIC / steady-state Vlasov simulations, and asymptotic analysis, on cylindrical Langmuir probes with $e\Phi_P/kT >> 1$, at rest in unmagnetized plasmas,

Electric potential and attracted-species density exhibit complex radial profiles; the density presents a maximum near the probe and a minimum well within the sheath.

Excellent agreement for probe radius R close to the maximum radius R_{max} for orbital-motion-limited (OML) collection at bias ratio ~ 5000, in several profile features: values and positions of density minimum and maximum, position of sheath boundary, and radius r_{eos} characterizing the no-space-charge behaviour of the potential near the high-bias probe.

Good agreement for parametric laws on i) sheath boundary versus R and Φ_P, for $R << R_{max}$; ii) on density minimum versus Φ_P for $R \cong R_{max}$; and iii) weakly Φ_P-dependent current drop below the OML value versus R for $R > R_{max}$.

Bare-Tether Sheath and Current: Comparison of Asymptotic Theory

and Kinetic Simulations **(submitted)**

J.R. Sanmartín, ***Universidad Politécnica de Madrid,***

E. Choinière, B.E. Gilchrist, ***University of Michigan***

J-B. Ferry, M. Martínez-Sánchez, ***Massachusetts Institute of Technology***

E. Choinière, B.E. Gilchrist, **<u>IEEE Trans. in Plasma Sci</u>., 35, pp. 7-22, Feb. 2007**

J-B. Ferry, **<u>M. S. Dissertation, MIT</u>, Cambridge, MA, June 2003**

J.R. Sanmartín and R.D. Estes, <u>Phys. Plasmas</u>, 6, pp. 395-405, Jan. 1999

R.D. Estes and J.R. Sanmartín, <u>Phys. Plasmas</u>, 7, pp. 395-405, Oct. 2000

Basics **Dimensionless magnitudes are functions of temperature T_i/T_e (= 1 here),**

bias $\bar{\Phi}_P \equiv e\Phi_P/kT >> 1$, and Debye $\bar{R} \equiv R/\lambda_D$ ratios

*** To find profiles** $\bar{n} \equiv N_e / N_\infty$, $\bar{\Phi} \equiv e\Phi / kT$ ***versus*** $\bar{r} \equiv r / \lambda_D$,

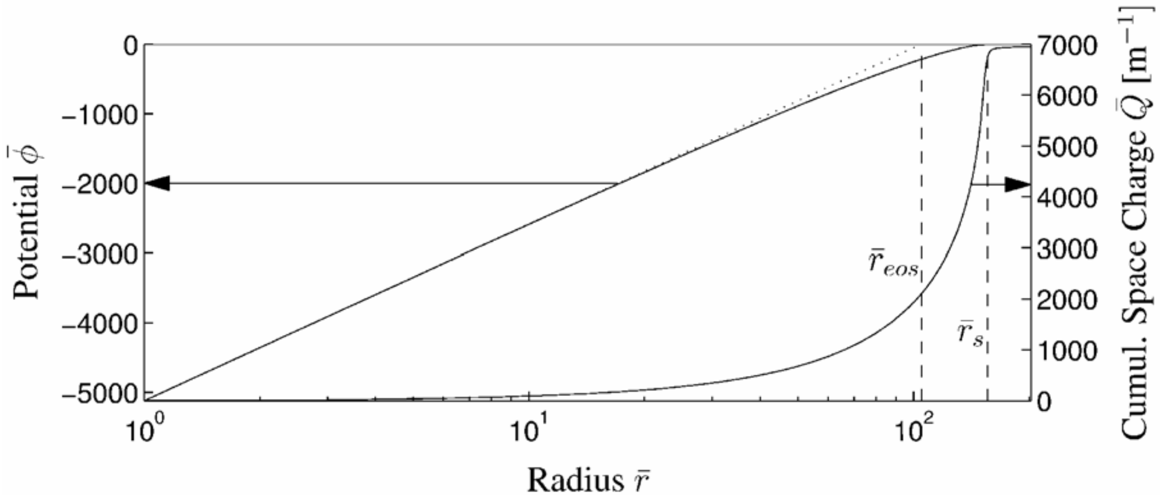
and normalized current I_e / I_{th} (I_{th} = random current), at given $\bar{\Phi}_P$ and $\bar{R}$,

Poisson´s equation must be solved jointly with the steady Vlasov equation

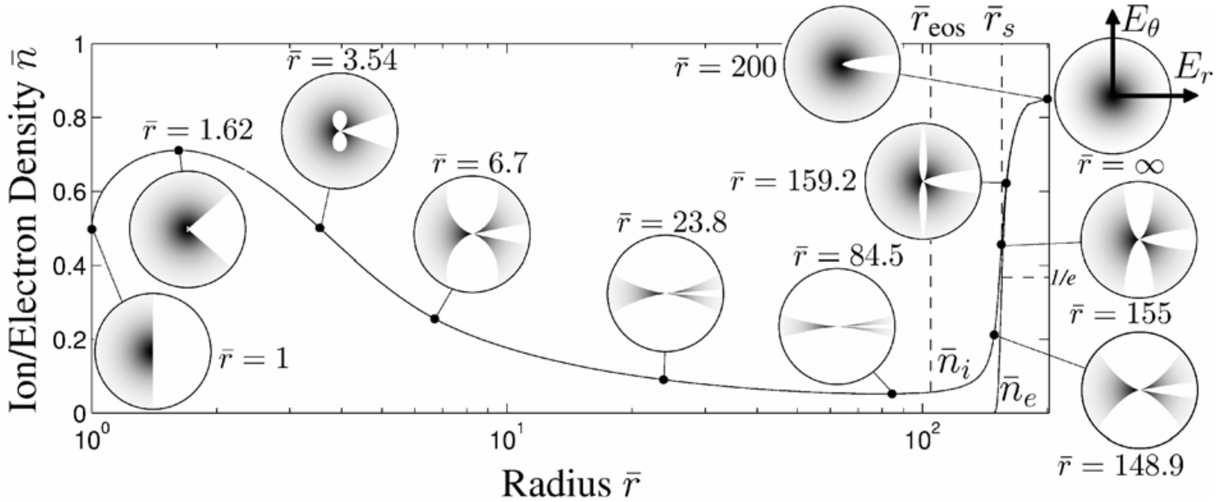
$$\frac{\lambda_D^2}{r}\frac{d}{dr}r\frac{d}{dr}\left(\frac{e\Phi}{kT}\right) = \frac{N_e}{N_\infty} - \exp\left(-\frac{e\Phi}{kT}\right) \qquad (\Phi = \Phi_P \text{ at } r = R, \quad \Phi \to 0 \text{ as } r \to \infty)$$

$$\vec{v}\cdot\nabla f_e - \frac{e}{m_e}\nabla\Phi\cdot\frac{\partial f_e}{\partial \vec{v}} = 0 \qquad \Rightarrow \qquad N_e = \int f_e\, d\vec{v} \qquad (f_e = f_M \text{ for } v_r < 0 \text{ as } r \to \infty)$$

<u>Numerical results</u> (at Univ. Michigan):



$\overline{R} = 1, \ \overline{\Phi}_P = 5120$



$\bar{r}(\bar{n}_{\max}) \approx 1.62, \qquad \bar{n}_{\max} \approx 0.71; \qquad \bar{r}(\bar{n}_{\min}) \approx 80.2, \qquad \bar{n}_{\min} \approx 0.052$

$$\bar{r}_s \equiv \bar{r}(\bar{\Phi} = 1) \approx 155.5 \qquad \bar{r}_{eos} \approx 104.8, \qquad \left[\frac{\Phi}{\Phi_p} \approx \frac{\ln(r_{eos}/r)}{\ln(r_{eos}/R)} \right] .$$

The Michigan code is a steady-state, self-consistent kinetic solver that simulates collisionless unmagnetized plasmas at rest in a vast region around a high-bias probe.

The Poisson solver uses the Finite Element Method. The Vlasov solver uses conservation of energy and angular momentum to infer f_e within the computational zone.

Key component of the solvers is an iterative approach to reaching a self-consistent solution based on successive linearizations of the nonlinear Poisson-Vlasov operator.

Tikhonov-regularized Newton iteration is used to handle numerical instabilities due to large grid sizes modeling a high-voltage large sheath. Sharp features encountered in net space charge near the sheath edge are resolved using an adaptive meshing strategy.

<u>Asymptotic analysis</u> The Vlasov equation conserves f_e: At a point r, $\bar{v}$,

$f_e = f_M \propto \exp(-E/kT)$ if its trajectory connects back to infinity, $f_e = 0$ if not

Electron trajectories conserve v_z, $J \equiv m_e r v_\theta$, $E \equiv \frac{1}{2} m_e v_r^2 + \frac{1}{2} m_e v_\theta^2 - e\Phi(r)$

$$n(r) = \int \frac{dE \exp(-E/kT_e)}{\pi kT} \int \frac{dJ}{\sqrt{J_r^2(E) - J^2}} .$$

$$J_r^2(E) - J^2 \equiv m_e^2 r^2 v_r^2 > 0, \qquad J_r^2(E) \equiv 2 m_e r^2 [E + e\Phi(r)]$$

Integrate over the range $0 \le E < \infty$ twice (for $v_r < 0$ <u>and</u> $v_r > 0$)

Condition $v_r'^2 > 0$ required throughout $r < r' < \infty$

J-range for $v_r < 0$: $0 < J < J_r^*(E) \equiv \text{minimum}\,[J_{r'}(E);\ r \le r' < \infty]$

J-range for $v_r > 0$: $J_R^*(E) < J < J_r^*(E)$ [Probe absorbs $J < J_R^*(E)$ electrons]

$$\Rightarrow \quad \bar{n}(r) = \int_0^\infty \frac{dE}{\sqrt{\pi kT_e}} \exp\left(-\frac{E}{kT_e}\right) \left\{ 2\sin^{-1}\left[\frac{J_r^*(E)}{J_r(E)}\right] - \sin^{-1}\left[\frac{J_R^*(E)}{J_r(E)}\right] \right\}$$

$0 < J < J_R$ (E) electrons make the current $\quad \Rightarrow \quad \frac{I_e}{I_{th}} = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dE}{kT_e} \exp\left(-\frac{E}{kT_e}\right) \frac{J_R^*(E)}{\sqrt{2m_e R^2 kT_e}}$

If $J_r^*(E) < J_r(E)$ $\Rightarrow$ potential barrier for r at $J_{r'}(E)$ minimum (at any E)

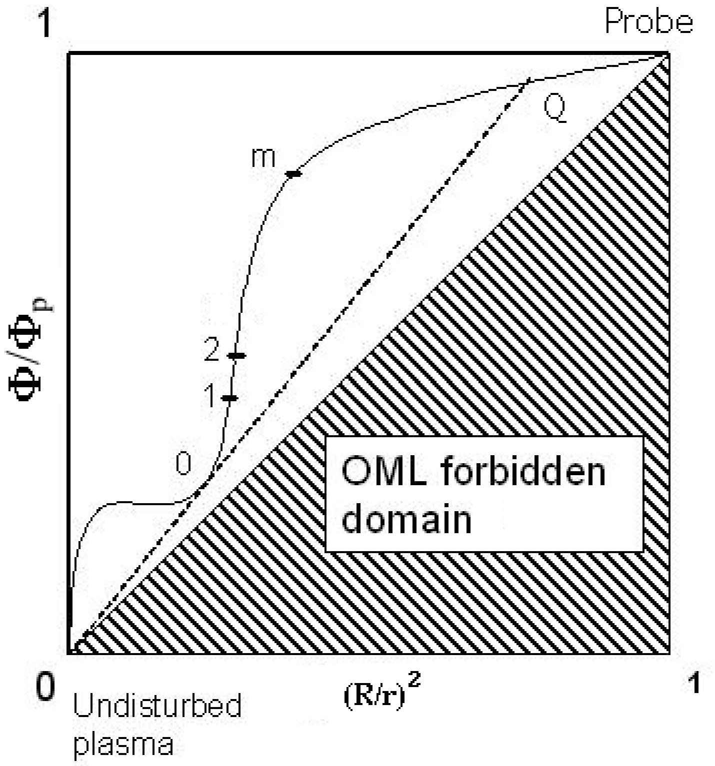
Maximum current (I_{OML}) if $J_R^*(E) \equiv J_R(E)$ [$\approx J_R(0)$ for $\bar{\Phi}_P >> 1$]

OML regime holds for $\bar{R} < \bar{R}_{max}(\bar{\Phi}_P, T_i/T_e = 1)$ $\left[\approx 1.09 \;\; for \;\; \Phi_P = 5120\right]$

No potential barrier for r = R <u>does not</u> imply no barrier $\forall$r

<u>No</u> barrier for r, $J_r^*(E) \equiv J_r(E)$ $\Rightarrow$ $r^2\Phi(r) \le r'^2\Phi(r')$, $r \le r' < \infty$

<u>No</u> barrier for R (OML regime) $\Rightarrow$ $\Phi_P R^2/r^2 \le \Phi(r)$ $R \le r < \infty$



Faraway quasineutrality

$\Rightarrow$ $\Phi \sim$ 1/r at large r

In a neighborhood of the probe, $\Phi \sim \ln r$

$r^2\Phi(r)$ decreases all the way to point 0,

increases up to point m, then drops to $R^2\Phi_P$

$J_r^*(E) \equiv J_r(E)$ for $r > r_0$

and from point Q to the probe

At $\overline{R} = \overline{R}_{max}$ point 0 lies on the diagonal.

Points 0, 1, 2 are near lower left corner

Quasineutrality fails at r_1 ($N_i \approx 0$ for $r < r_2$) **Within the sheath, $J_r(E) \approx J_r(0)$**

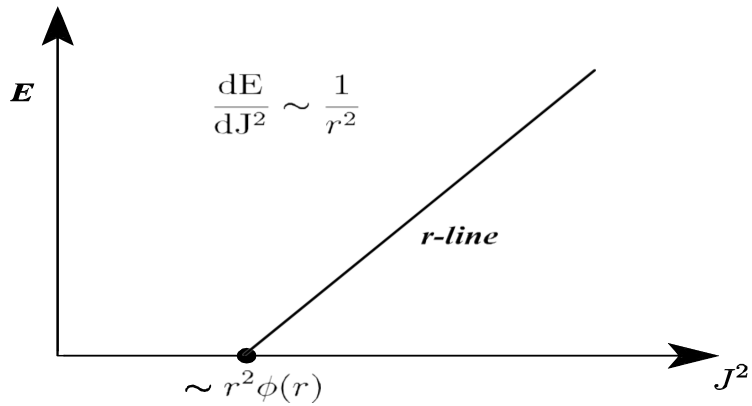
$J_r^*(E)$ determined by the graph Φ / Φ_P versus $(R/r)^2$

and the r-family of straight lines $J^2 = J_r^2(E) \equiv 2m_e r^2[E + e\Phi(r)]$

Slope $\sim 1/r^2$ steepens all the way from ∞ to R

Line-foot $J_r^2(0)$ moves left from ∞ to r_0 ,

right from r_0 to r_m, left from r_m to R



Lines $r_0 - r_1$ generate envelope $J_{env}(E)$ in terms of r_0, Φ_0,

$J_{env}(E)$ determined, in terms of the free parameter $r_0^2\,\Phi_0 / R^2\,\Phi_P$,

by quasineutrality conditions at r_0 and r_1,

and by derivating the quasineutrality relation at r_1 (where $dr/d\Phi = 0$)

Density maximum $J_{r*}(E) \equiv J_r(E)$ **does hold from point Q to the probe**

$$\pi(r) \;\approx\; 1 - \int_0^\infty \frac{dE\,\exp(-E/kT)}{\pi\, kT}\; \sin^{-1}\sqrt{\frac{R^2\Phi_p}{r^2\Phi}}\;.$$

If the density maximum was at Q, consistency checked by using $\left[\dfrac{\Phi}{\Phi_p} \approx \dfrac{\ln(r_{eos}/r)}{\ln(r_{eos}/R)}\right]$ **and**

setting $\bar{R}=1$, $\bar{r}_{eos}$ **= 104.8,** $\bar{r}_Q = \bar{r}(\bar{n}_{max})$ **= 1.62,**

$$\bar{n}_{max} = \bar{n}_Q = 1 - \frac{1}{\pi}\sin^{-1}\left[\frac{\bar{R}}{\bar{r}_Q}\sqrt{\frac{\ln(\bar{r}_{eos}/\bar{R})}{\ln(\bar{r}_{eos}/\bar{r}_Q)}}\right] \;\rightarrow\; \bar{n}_{max} \approx 0.77\,,$$

as against $\bar{n}_{max} \approx 0.71$ **in the simulations.**

Actually the maximum lies somewhat beyond Q, $\bar{r}_Q < \bar{r}(\bar{n}_{max}) = 1.62 \;\Rightarrow\; \bar{n}_{max} < 0.77$

Density minimum **Results used correspond to** $\overline{R} = \overline{R}_{\max} = 1.09$ **(Phys. Plasmas 1999)**

Within the sheath, away from the probe,

$$\mathbf{J_r^*(E) = J_{env}(E) << J_r(E) \approx J_r(0), \quad J_R^*(E) \approx J_R(0)}$$

$$\Rightarrow \qquad \underline{n} \approx \left\{ \int_0^\infty \frac{dE}{\pi\, kT_e} \exp\left(\frac{-E}{kT_e} \right) \left[\frac{2J_{env}(E)}{J_R(0)} - 1 \right] \right\} \times \sqrt{\frac{R^2 \Phi_P}{r^2 \Phi(r)}} \approx \frac{3.40}{\pi} \sqrt{\frac{R^2 \Phi_P}{r^2 \Phi(r)}}$$

Define u ≡ ln (r_2 / r) with r_2 at sheath boundary, g ≡ properly normalised Φ

$$\Rightarrow \qquad \mathbf{d^2g/du^2 = \exp(-u)/\sqrt{g}, \qquad [g = dg/du = 0 \qquad \text{at} \quad u = 0]}$$

In a no space–charge neighborhood of the probe (large u)

$$\mathbf{g(u) \approx c\,(u - b), \qquad c \approx 2.09, \qquad b \approx 0.351}$$

$$\frac{\Phi}{\Phi_p} \approx \frac{\ln(r_2/r) - b}{\ln(r_2/R) - b} \qquad \Rightarrow \qquad r_{eos} = r_2\, e^{-b}.$$

Further in the sheath $r^2\Phi(r) \propto e^{-2u}\, g(u)$ has a maximum at point m ($u_m = 0.63$, $g_m = 0.86$)

$$r(n_{min}) = r_2\, e^{-0.63} \qquad \Rightarrow \qquad r(n_{\min})/r_{eos} = e^{0.351-0.63} \approx 0.756,$$

as compared with a ratio 0.765 from the simulations.

<u>The density minimum</u> itself is

$$n_{\min} = \frac{3.40}{\pi} \frac{R}{r_2} \sqrt{\frac{\exp(2u_m)}{g_m} c \left(\ln \frac{r_2}{R} - b \right)} \approx 0.049$$ **(0.052 in the simulations)**

using results for R / r_1 and r_1 / r_2 shown below

<u>Sheath boundary</u> **The theoretical analysis gives** $\overline{\Phi}_1$ **= O(1) < 1 and** $\overline{\Phi}_2$ **>> 1, at the points 1 and 2, which lie very close to each other at high bias** $\Rightarrow$ $\mathbf{r_s \approx r_1}$

$$\frac{r_2}{r_1} = 1 - \frac{1.62}{\overline{\Phi}_P^{\,2/5}\,\overline{R}_{\max}^{\,4/5}} \approx 0.95$$

$$r_s \approx r_1 = \frac{r_1}{r_2} r_2 \qquad \Rightarrow \qquad \frac{r_s}{r_{eos}} \approx \frac{r_1}{r_2} \exp(b) \approx 1.490,$$ **(1.484 in the simulations)**

Also, point 1 is shown to obey a relation $\overline{\Phi}_p R^2 / r_1^2 \approx 0.24$ **(R = R$_{\mathbf{max}}$, T$_{\mathbf{i}}$ = T$_{\mathbf{e}}$)**

$$\Rightarrow \quad \overline{r}_s \approx \overline{R}_{\max} \frac{r_1}{R_{\max}} \approx 1.09 \times \sqrt{\frac{1520}{0.24}} \approx 159.2$$ **(155.5 in the simulations)**

Sheath-boundary law for R << R_{max}

A fitted law from steady-Vlasov simulations

$$(\bar{R} = 0.001, \quad \bar{r}_s = 10-80)$$

$$\Rightarrow \quad 1.298\, \bar{r}_s^{\,1.346} \ln\left(\frac{\bar{r}_s}{\bar{R}}\right) \approx \bar{\Phi}_P$$

$\bar{\Phi}_P$ **values are only moderately large**

Asymptotic analysis

$$\Rightarrow \quad 1.53 \left[1 - \frac{2.56}{\bar{r}_s^{\,4/5}}\right] \bar{r}_s^{\,4/3} \ln\left(\frac{\bar{r}_s}{\bar{R}}\right) \approx \bar{\Phi}_P$$

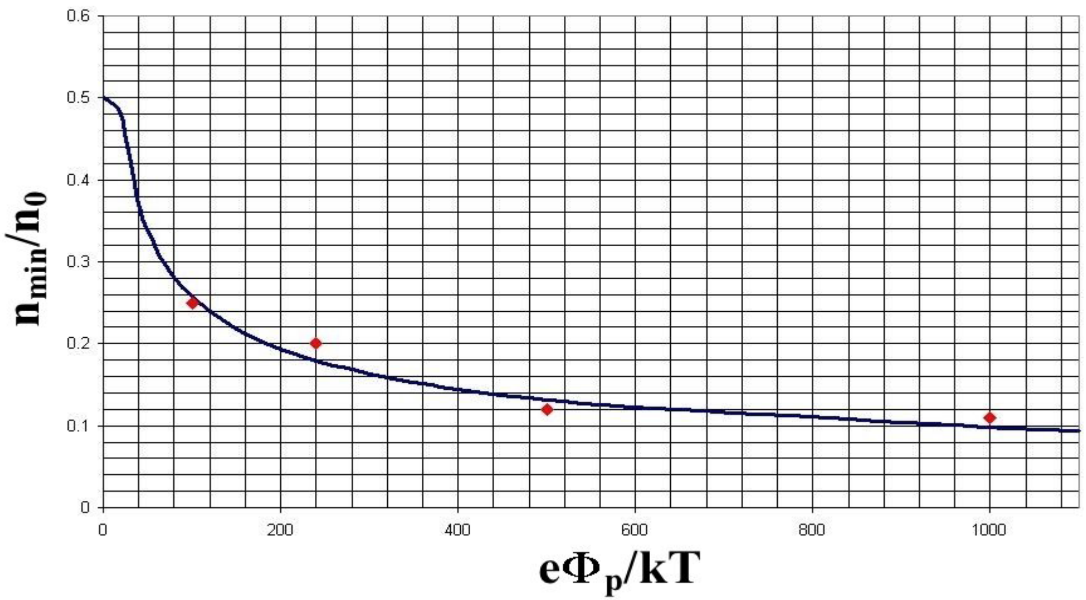
Good agreement except at the lowest $\bar{r}_s$ **values, corresponding to lowest** $\bar{\Phi}_P$,

the asymptotic analysis finally breaking down

<u>Density minimum law for</u> R = R$_{max}$ (T$_i$ = T$_e$)

$$\bar{n}_{min}(\bar{\Phi}_p) = 1.097\,\frac{r_1}{r_2}\sqrt{\frac{\ln\bar{\Phi}_p + 0.725 - 2\ln(r_1/r_2)}{\bar{\Phi}_p}}\,, \qquad \frac{r_2}{r_1} = 1 - \frac{1.62}{\bar{\Phi}_p^{2/5}\,\bar{R}_{max}^{4/5}}$$

Comparison to (R = λ_D) PIC simulations at MIT



Probe current law for $R > R_{max}$

$$I / I_{OML} = G\left[\overline{\Phi}_P,\ \overline{R},\ T_i / T_e = 1\right], \qquad \overline{R} > \overline{R}_{max}(\overline{\Phi}_P,\ T_i / T_e = 1)$$

A few percent differences with PIC simulations (ignoring weak dependence on Φ_P)